Laplace Transforms

Transform methods in differential equations are very general. There are Laplace transforms, Fourier transforms, Hankel transforms, Legendre transforms, etc. Each type of transform is specific to solving a particular type of IVP or BVP with either an ODE or PDE. Laplace transforms are used to solve an IVP with an ODE. We will solve linear ODE's with constant coefficients and initial conditions, i.e. problems of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(t), \quad \text{where } y = y(t)$$

$$y(0) = y_0$$
$$y'(0) = y_0'$$
$$\vdots$$
$$y^{(n-1)}(0) = y_0^{(n-1)}$$

The general idea of all transform methods is to convert a complicated expression into a simpler expression which can be solved. Then the original quantity of interest is recovered by an inverse operation.

Ex: This is a silly example of the method using functions but it illustrates the procedure.

Suppose you want to find the product of two reals,

$$3.163 \times 16.380$$

but you don't know how to mutiply. You could transform this expression by the use of the logarithm –

$$\ln(3.163 \times 16.380) = \ln(3.163) + \ln(16.380)$$
$$= 1.152 + 2.796$$
$$= 3.948$$

Notice what happened. We transformed the operation of multiplication into the operation of addition by use of the logarithm. To recover the answer to the original question we make use of the inverse of the logarithm, namely the exponential function –

$$e^{\ln(3.163 \times 16.380)} = 3.163 \times 16.380$$
$$= e^{3.948}$$
$$\boxed{= 51.832}$$

Transform methods are similar. Instead of transforming an operation (multiplication) by using a function (the logarithm) into another operation (addition) and then recovering the solution by use of the inverse function we transform an IVP using the Laplace transform into an algebraic equation which we solve for the transformed function. We then recover the solution to the IVP by use of the inverse transform.

So let's define the Laplace transform of $f(t)$ —

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s) \quad , s > 0$$

Some things to notice —

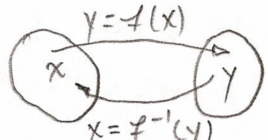
1) we use the corresponding capital letter to denote the Laplace transform of a function of t. i.e. $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$, etc....
2) Technically, s is a complex variable but for our purposes that's irrelevant. Just consider s to be a dummy variable, it won't matter.

We then define the inverse Laplace transform, $\mathcal{L}^{-1}$, in terms of the Laplace transform —

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \iff \mathcal{L}\{f(t)\} = F(s)$$

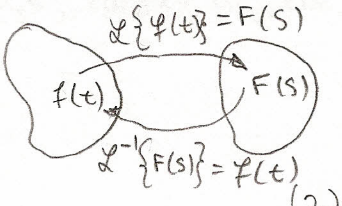
So we can not directly calculate $\mathcal{L}^{-1}\{F(s)\}$ (it is an integral in complex plane) but for every Laplace transform we calculate we have an inverse transform for free.

This is analogous to the definition of an inverse function, if $y = f(x)$ then $x = f^{-1}(y) \iff y = f(x)$.



So $f(x)$ maps the set of x into the set y and f^{-1} is the inverse map from y to x. Then we know $f(f^{-1}(y)) = f(x) = y$ and $f^{-1}(f(x)) = f^{-1}(y) = x$.

The Laplace transform is essentially the same but, instead of being a function of a real variable, the transform is a function of a function $f(t)$. $\mathcal{L}$ maps the set of functions $f(t)$ to the set of functions $F(s)$ and $\mathcal{L}^{-1}$ is the inverse mapping.



So $\mathcal{L}^{-1}(\mathcal{L}\{f(t)\}) = \mathcal{L}^{-1}\{F(s)\} = f(t)$

and $\mathcal{L}(\mathcal{L}^{-1}\{F(s)\}) = \mathcal{L}\{f(t)\} = F(s)$

which look just like the properties of functions and their inverses.

The last useful property we need is that both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear operators,

i.e. $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$

and $\mathcal{L}^{-1}\{c_1 F(s) + c_2 G(s)\} = c_1 \mathcal{L}^{-1}\{F(s)\} + c_2 \mathcal{L}^{-1}\{G(s)\}$

Proof: The first property simply follows from linearity of the integral –

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}(c_1 f(t) + c_2 g(t))\,dt$$
$$= c_1 \int_0^\infty e^{-st} f(t)\,dt + c_2 \int_0^\infty e^{-st} g(t)\,dt$$
$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\} \quad //$$

Linearity of $\mathcal{L}^{-1}$ follows directly –

$$\mathcal{L}^{-1}\{c_1 F(s) + c_2 G(s)\} = \mathcal{L}^{-1}\{c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}\}$$
$$= \mathcal{L}^{-1}\{\mathcal{L}\{c_1 f(t) + c_2 g(t)\}\}$$
$$= c_1 f(t) + c_2 g(t)$$
$$= c_1 \mathcal{L}^{-1}\{F(s)\} + c_2 \mathcal{L}^{-1}\{G(s)\} \quad //$$

Let's calculate some transforms. You won't have to do this every time, transform pairs are collected in tables. A good table is attached, I will refer to equation numbers in that table.

Ex: Find $\mathcal{L}\{1\}$. Here $f(t) = 1$ so

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt$$
$$= \lim_{T\to\infty} \int_0^T e^{-st}\,dt$$
$$= \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^T$$
$$= \lim_{T\to\infty} \left(-\frac{1}{s} e^{-sT} + \frac{1}{s} e^0\right)$$
$$= 0 + \frac{1}{s}$$
$$= \boxed{\frac{1}{s}}$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ and, by definition of the inverse transform, we also know

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1.$$

This is #1 in the attached table.

Ex: Find $\mathcal{L}\{e^{at}\}$ –

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\,dt$$

$$= \lim_{T\to\infty} \left[-\frac{1}{s} e^{-(s-a)t}\right]_0^T$$

$$= \lim_{T\to\infty} \left[\frac{-1}{s-a} e^{-(s-a)T} + \frac{1}{s-a} e^0\right]$$

$$= \frac{1}{s-a} \quad \text{if } a < s$$

Also we know, by definition, that

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

This transform pair is #2 in our table.

Notice in the above that $\mathcal{L}\{e^{at}\} = \infty$ which means it is undefined if $a > s$. This suggests not every function has a Laplace transform.

Theorem: If $|f(t)| \le Me^{kt}$ for some constants M, k where $k < s$, then $\mathcal{L}\{f(t)\}$ exists.

Proof: This is easy to see –

$$\mathcal{L}\{Me^{kt}\} = \int_0^\infty e^{-st} \cdot Me^{kt}\,dt$$

$$= M \int_0^\infty e^{-(s-k)t}\,dt$$

$$= M \left[\frac{-1}{s-k} e^{-(s-k)t}\right]_0^\infty$$

$$= \begin{cases} \frac{M}{s-k}, & k < s \\ \infty, & k > s \end{cases}$$

So as long as $|f(t)| < Me^{kt}$, $k < s$ then the integral converges and $\mathcal{L}\{f(t)\}$ exists. Notice this means that things like $\mathcal{L}\{t^t\}$ and $\mathcal{L}\{t!\}$ do not exist. In physical problems this won't matter because these functions won't occur.

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ex: Find $\mathcal{L}\{t\}$ -

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st} dt$$

This is by parts where

$$u = t, \quad dv = e^{-st} dt$$
$$du = dt, \quad v = -\frac{1}{s} e^{-st}$$

so
$$\int t e^{-st} dt = -\frac{t}{s} e^{-st} + \frac{1}{s} \int e^{-st} dt$$
$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}$$

So
$$\mathcal{L}\{t\} = \lim_{T \to \infty} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^T$$
$$= \lim_{T \to \infty} \left(-\frac{T}{s} e^{-sT} - \frac{1}{s^2} e^{-sT} + 0 + \frac{1}{s^2} e^0 \right)$$
$$= \frac{1}{s^2}$$

So $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ (#3 in table with $n = 1$)

#3, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ is straightforward to see by deduction.

Consider $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2 dt$

Let $u = t^2$, $dv = e^{-st} dt$
$du = 2t\,dt$, $v = -\frac{1}{s} e^{-st}$

So
$$\mathcal{L}\{t^2\} = \left[-\frac{t^2}{s} e^{-st} + \frac{2}{s} \int t e^{-st} dt \right]_0^\infty$$
$$= \frac{2}{s} \int_0^\infty t e^{-st} dt$$
$$= \frac{2}{s} \mathcal{L}\{t\}$$
$$= \frac{2}{s} \cdot \frac{1}{s^2}$$
$$= \frac{2!}{s^3}$$

and, $\mathcal{L}\{t^3\} = \int_0^\infty t^3 e^{-st} dt$ $\quad u = t^3$, $dv = e^{-st} dt$, $du = 3t^2 dt$, $v = -\frac{1}{s} e^{-st}$

$$= \left[-\frac{t^3}{s} e^{-st} + \frac{3}{s} \int t^2 e^{-st} dt \right]_0^\infty$$
$$= \frac{3}{s} \int_0^\infty t^2 e^{-st} dt$$
$$= \frac{3}{s} \mathcal{L}\{t^2\}$$
$$= \frac{3}{s} \cdot \frac{2!}{s^3}$$
$$= \frac{3!}{s^4}$$

You can continue for any n.

You can find $\mathcal{L}\{\cos\omega t\}$ and $\mathcal{L}\{\sin\omega t\}$ directly, for example –

$$\mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-st}\sin\omega t\,dt$$

Recall this is by parts twice –

$$u = \sin\omega t, \quad dv = e^{-st}dt$$
$$du = \omega\cos\omega t\,dt, \quad v = -\tfrac{1}{s}e^{-st}$$

Then $\int e^{-st}\sin\omega t\,dt = -\frac{1}{s}\sin\omega t\, e^{-st} + \frac{\omega}{s}\int \cos\omega t\, e^{-st}dt$

This remaining integral is again by parts –

$$u = \cos\omega t, \quad dv = e^{-st}dt$$
$$du = -\omega\sin\omega t, \quad v = -\tfrac{1}{s}e^{-st}$$

So

$$\int e^{-st}\sin\omega t\,dt = -\frac{1}{s}\sin\omega t\, e^{-st} + \frac{\omega}{s}\left\{-\frac{1}{s}\cos\omega t\, e^{-st} - \frac{\omega}{s}\int \sin\omega t\, e^{-st}dt\right\}$$

$$\left(1+\frac{\omega^2}{s^2}\right)\int e^{-st}\sin\omega t\,dt = -\frac{1}{s}\sin\omega t\, e^{-st} - \frac{\omega}{s^2}\cos\omega t\, e^{-st}$$

and

$$\int e^{-st}\sin\omega t\,dt = \frac{1}{1+\frac{\omega^2}{s^2}}\left\{-\frac{1}{s}\sin\omega t\, e^{-st} - \frac{\omega}{s^2}\cos\omega t\, e^{-st}\right\}$$

Whew!!

So $\mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-st}\sin\omega t\,dt$

$$= \frac{s^2}{s^2+\omega^2}\cdot\lim_{T\to\infty}\left[-\frac{1}{s}\sin\omega t\, e^{-st} - \frac{\omega}{s^2}\cos\omega t\, e^{-st}\right]_0^T$$

$$= \frac{s^2}{s^2+\omega^2}\left(-0-0+0+\frac{\omega}{s^2}\right)$$

$$= \frac{\omega}{s^2+\omega^2}$$

So we have the transform pair –

$$\left.\begin{aligned} &\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} \\ &\text{and } \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t \end{aligned}\right\} \#7 \text{ in table}$$

That was fun but it seems like a lot of work.
There is a shortcut!

We can find $\mathcal{L}\{\cos\omega t\}$ and $\mathcal{L}\{\sin\omega t\}$ simultaneously –

Consider Euler's formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$.

Then $\mathcal{L}\{e^{i\omega t}\} = \frac{1}{s-i\omega}$ from previous example (table #2)

$$= \frac{1}{s-i\omega}\cdot\frac{s+i\omega}{s+i\omega}$$

$$= \frac{s+i\omega}{s^2+\omega^2}$$

$$= \frac{s}{s^2+\omega^2} + i\,\frac{\omega}{s^2+\omega^2}$$

So $\mathcal{L}\{e^{i\omega t}\} = \mathcal{L}\{\cos\omega t + i\sin\omega t\}$

$= \mathcal{L}\{\cos\omega t\} + i\mathcal{L}\{\sin\omega t\}$ (by linearity of Laplace transform)

$$= \frac{s}{s^2+\omega^2} + i\,\frac{\omega}{s^2+\omega^2}$$

Equating real and imaginary parts we have –

$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$ (table #7 and #8)

That seems easier!

Shift theorem –

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ (table #29)

Proof: $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$ by definition of $\mathcal{L}$

$$= \int_0^\infty e^{-st}\left(e^{at}f(t)\right)dt$$

$$= \mathcal{L}\{e^{at}f(t)\} \quad //$$

This is called a shift theorem because multiplication of $f(t)$ by e^{at} shifts the argument of the transform by a, analogous to the shift of the graph of $y = f(x-a)$ by a.

As examples – we know

$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$ (#8) so $\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$ (#20)

$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$ (#7) so $\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$ (#19)

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ (#3) so $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$ (#23)

etc...

Transform of derivatives

This is the key to solving the IVP. The proof of the derivative of any order is by induction.

Consider $\mathcal{L}\{f'(t)\} = \mathcal{L}\{\frac{df}{dt}\}$

$$= \int_0^\infty e^{-st} f'(t)\,dt$$

This is by parts where

$$u = e^{-st}, \quad dv = f'(t)\,dt$$

$$du = -se^{-st}dt, \quad v = f(t)$$

So $\mathcal{L}\{f'(t)\} = \left[e^{-st}f(t) + s\int e^{-st}f(t)\,dt\right]_0^\infty$

$$= 0 - f(0) + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= sF(s) - f(0)$$

Notice that we can express $\mathcal{L}\{f'(t)\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$ <u>and</u> the initial condition $f(0)$. This is key to solving the IVP.

Next consider $\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$

again by parts where $u = e^{-st}, \quad dv = f''(t)\,dt$

$$du = -se^{-st}dt, \quad v = f'(t)$$

Then $\mathcal{L}\{f''(t)\} = \left[e^{-st}f'(t) + s\int e^{-st}f'(t)\,dt\right]_0^\infty$

$$= 0 - f'(0) + s\int_0^\infty e^{-st}f'(t)\,dt$$

$$= s\mathcal{L}\{f'(t)\} - f'(0)$$

$= s\{sF(s) - f(0)\} - f'(0)$ from result above for $\mathcal{L}\{f'(t)\}$

$$= s^2F(s) - sf(0) - f'(0)$$

And we have $\mathcal{L}\{f''(t)\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$ and two initial conditions $f(0)$ and $f'(0)$.

Let's do one more to make sure we see the pattern –

$\mathcal{L}\{f'''(t)\} = \int_0^\infty e^{-st} f'''(t)\,dt$

by parts – $u = e^{-st}$, $dv = f'''(t)\,dt$

$du = -se^{-st}dt$, $v = f''(t)$

So $\mathcal{L}\{f'''(t)\} = \left[e^{-st}f''(t) + s\int e^{-st}f''(t)\,dt\right]_0^\infty$

$= 0 - f''(0) + s\int_0^\infty e^{-st}f''(t)\,dt$

$= s\mathcal{L}\{f''(t)\} - f''(0)$

$= s\{s^2F(s) - sf(0) - f'(0)\} - f''(0)$ from result for $\mathcal{L}\{f''(t)\}$

$= s^2F(s) - s^2f(0) - sf'(0) - f''(0)$

and, in general –

$$\mathcal{L}\{f^{(n)}(t)\} = \mathcal{L}\left\{\frac{d^nf}{dx^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - s^2f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

This is #37 in the table. The pattern is simple. The first term is $s^nF(s)$, all remaining terms are negative, lower the power of s by one and raise the derivative by one until s vanishes and/or the last term is $f^{(n-1)}(0)$.

Now that we can express the Laplace transform of any derivative of $f(t)$ in terms of the transform, $F(s)$, of $f(t)$ and initial conditions on $f(t)$ we can solve an IVP. The process is to transform the equation to find an equation in $F(s)$ and the initial conditions (called the subsidiary equation), solve this subsidiary equation for $F(s)$, then take the inverse transform of $F(s)$ to recover $f(t)$. So let's solve an IVP –

Ex: Solve $y'' - y = t$, $y(0) = y'(0) = 1$ where $y = y(t)$

First we transform both sides of the equation –

$\mathcal{L}\{y'' - y\} = \mathcal{L}\{t\}$

$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{t\}$ by linearity of the Laplace transform

$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s^2}$ by #3 and #37 where $\mathcal{L}\{y(t)\} = Y(s)$

This is the subsidiary equation in $Y(s)$. It is purely algebraic, there are no derivatives, we can always solve it.

Now substitute the initial conditions $y(0) = y'(0) = 1$

$$s^2 Y(s) - s(1) - 1 - Y(s) = \frac{1}{s^2}$$

$$(s^2 - 1)Y(s) - s - 1 = \frac{1}{s^2}$$

now solve for $Y(s)$ –

$$(s^2-1)Y(s) = \frac{1}{s^2} + s + 1$$

$$Y(s) = \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1}$$

$$= \frac{1}{s^2(s^2-1)} + \frac{1}{s-1}$$

since $\mathcal{L}^{-1}\{Y(s)\} = y(t)$ we can recover our solution. The second term on the RHS, $\frac{1}{s-1}$, appears in our table. The first term requires partial fractions –

$$\frac{1}{s^2(s^2-1)} = \frac{1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1}$$

$$1 = As(s+1)(s-1) + B(s+1)(s-1) + Cs^2(s-1) + Ds^2(s+1)$$

then $A = 0$

$1 = -B$

$B = -1$

$1 = 2D$

$D = \frac{1}{2}$

$1 = -2C$

$C = -\frac{1}{2}$

so $Y(s) = \frac{-1}{s^2} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{s-1}$

$$Y(s) = \frac{3}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{s^2}$$

Now we take the inverse transform –

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{s^2}\right\}$$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$ by linearity of $\mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} - t}$$ by # 2 and # 3

or by # 29 and # 3

You probably noticed you can solve this by undetermined coefficients, I think it might be less work. You should do that and check the answer.

ex: Solve $y''+2y'+y=e^{-t}$, $y(0)=-1$, $y'(0)=1$

$$\mathcal{L}\{y''+2y'+y\}=\mathcal{L}\{e^{-t}\}$$

$$\left(s^2Y(s)-sy(0)-y'(0)\right)+2\left(sY(s)-y(0)\right)+Y(s)=\frac{1}{s+1}$$ by linearity, #37, #2

$$s^2Y(s)+s-1+2sY(s)+2+Y(s)=\frac{1}{s+1}$$ from initial conditions

$$(s^2+2s+1)Y(s)=\frac{1}{s+1}-s-1$$

$$(s+1)^2Y(s)=\frac{1}{s+1}-(s+1)$$

$$Y(s)=\frac{1}{(s+1)^3}-\frac{1}{s+1}$$

Now recover $y(t)$ from the inverse transform –

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}-\frac{1}{s+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$ by linearity of $\mathcal{L}^{-1}$

The first term is $\frac{1}{s^3}$ shifted by -1, the second term is $\frac{1}{s}$ shifted by -1.

So $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}$ by #2 or by #3 and #29

For the other term notice

$$\mathcal{L}\{t^2\}=\frac{2!}{s^3}=\frac{2}{s^3}$$ from #3

So $\frac{1}{s^3}=\mathcal{L}\left\{\frac{t^2}{2}\right\}$

and $\frac{1}{(s+1)^3}=\mathcal{L}\left\{e^{-t}\cdot\frac{t^2}{2}\right\}$ by #29

or directly from #23 –

$$\mathcal{L}\{t^2e^{-t}\}=\frac{2!}{(s+1)^3}=\frac{2}{(s+1)^3}$$

so $\mathcal{L}\left\{\frac{1}{2}t^2e^{-t}\right\}=\frac{1}{(s+1)^3}$

Either way we have

$$y(t)=\frac{1}{2}t^2e^{-t}-e^{-t}$$

$$\boxed{y(t)=e^{-t}\left(\frac{1}{2}t^2-1\right)}$$

Conclusion –

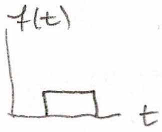
Since this is an introductory course this is as far as I want to go but you should be aware of some things.

1) This topic gets complicated, you can take a full semester graduate course on Transforms but all transforms work the same whether it is an IVP or BVP with an ODE or PDE. Each kind of transform applies to one case but you transform the equation to find a subsidiary equation, plug in the conditions, solve for the transformed equation, then take the inverse transform to recover the function of interest.

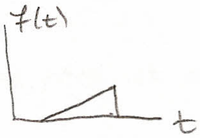
2) Although we only solved 2^{nd} order equations, this method applies to any order by the use of #37 in your table. The other method we know requires you to factor a characteristic equation of degree n which is in general impossible so if you have a 3^{rd} or higher order equation the method of Laplace transforms might be your only choice.

3) We could have used undetermined coefficients in our examples (you try it). The real utility of Laplace transforms is with strange non-homogeneous terms $f(t)$. For example you can have –

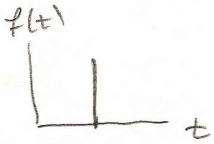
pulses



ramps



spikes



These complicate problems considerably and we would need weeks to discuss them properly. I'm not going down that rabbit hole. These occur in electrical engineering and if that is your interest you will learn how to solve these problems in your EE classes or a mathematical methods course. If your interest is not EE you probably won't ever see these types of signals so for most of you it's inappropriate to spend our class time on such.